Numerical evaluation for two-dimensional integral using higher order Gaussian quadrature

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Abstract: Presented in this paper is a computational approach that uses higher order Gaussian quadrature to improve the accuracy of the evaluation of an integral. The transformation from $\xi\eta$ space (standard Gaussian) to st space (higher order Gaussian) were shown throughout this paper. Not even that, the efficacy of this higher order Gaussian quadrature were tested by implementing and comparing it with standard Gaussian quadrature over the same integral. Results shown that the evaluation of an integral by using higher order Gaussian quadrature provide accurate and converge results compared to an integral using standard Gaussian quadrature.

Keywords: Gaussian quadrature; Integral; Rational Functions

INTRODUCTION

There are numerous numerical integration methods such as Newton-Cotes formula, Trapezoidal Rule, Simpson Rule as well as Gaussian quadrature. Among all these numerical integration technique, Gaussian quadrature is well known as the method for its accurate approximation of an integral over a domain (Hussain, Karim,& Ahamad, 2012). The algorithm is applicable to a wide range of functions including smooth functions as well as functions containing singularities. This type of numerical integration has been used by many researchers to cope with different types of integral. Some of the researchers used Gaussian quadrature formulas to evaluate the integrals contain singular and nearly singular (Graglia & Lombardi, 2008; Kaneko & Xu, 1994; Ma, Rokhlin,& Wandzura, 1996). Moreover, Gaussian quadrature has also been implemented in an integral containing arbitrary function and stochastic differential equations (Kloeden & Shardlow, 2017; Monegato & Scuderi, 2005). Gaussian quadrature rules has also been employed in study of solar-irradiance spectrum where they computed the integral over the wavelength (Johnson, 2019). Not even that, Gaussian quadrature is prominent because of its effectiveness in dealing with one-dimensional integrals containing smooth functions as stated in (Butler & Moffitt, 1982). This integrals was then extended to two-dimensional case. For an integral containing function f, the solution of the integral is represented and evaluated using the weighting sum. Other approach when dealing with numerical integration with singular integrand had been presented by (Schwartz, 1969). They started the evaluation with the consideration of Euler-Maclaurin sum formula. In order to decrease the error under the integration signs, they used a change of variables in their functions. However, this Euler-Maclaurin prediction seemed to be not reliable (Schwartz, 1969). They also stated that, when dealing with 1 numerical integration, the most efficient methods be used was the form of Gaussian quadrature. So, after applying the Euler-Maclaurin formula, they used the Gaussian quadrature form to get convergence results.

At n-point of quadrature rule, Gaussian quadrature is represented as the following equation in one-dimensional where the weight function is denoted by $w(x)$ and the approximation is exact whenever function f is a polynomial of degree less than 2n – 1.

$$\int_a^b f(x)dx \approx \sum_{r=1}^n w_r f(x_r)$$

where a and b is the limit of the integration.

The intention of this paper is to provide the numerical approximation for two-dimensional integrals containing rational functions. The extension from the standard triangular domain of $\varepsilon\eta$ in (Hussain, Karim,& Ahamad, 2012) into square domain of st domain will be shown throughout this paper. In order to obtain more accurate results, the weight and nodes of Gaussian quadrature

will be increase. From this transformation, we will then compared the numerical results for both approach. This paper is organized by following manner. Next section will describe the algorithm of the higher order Gaussian quadrature while Section 3 provide all the numerical approximation of an integral with known analytical solution where we recorded it in table form. Lastly, a summary and the concluding marks on the results and the approach used are given in Section 4.

METHOD

We start the methodology of higher order Gaussian quadrature by transforming standard Gaussian quadrature which is in triangular form of (ξ, η) space where $0 \le \xi \le 1$, and $0 \le \eta \le 1$ to square space which is (s, t) space where $-1 \le s \le 1$ and $-1 \le t \le 1$. Figure 1 showed the transformation of the triangle in $\xi\eta$ space into square in st space.
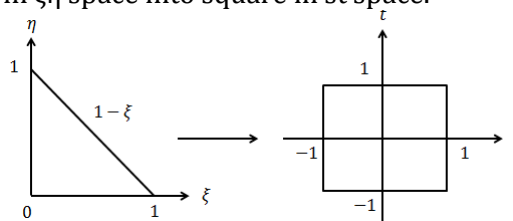

Figure 1: The transformation from triangular element ($\xi\eta$ space) into square element (st space)

We refer to the equation containing ξ and η as in equation (1)

$$I_1 = \iint f(x, y)dxdy \tag{1}$$

$$= \int_0^1 \int_0^{1=\xi} f(\xi, \eta) \,|Jac|d\xi d\eta$$

We change equation in (1) by transforming the ξ and η to s and t by substituting;

$$\xi = (1 - \frac{1+t}{2})(\frac{1+s}{2}), \tag{2}$$

$$\eta = \frac{1+t}{2}$$

where we change the shape functions of ξ and η to new shape functions containing s and t. For the next step, by finding the Jacobian in s and t terms, we will obtain;

$$\frac{d\xi}{ds}\frac{d\eta}{dt} - \frac{d\xi}{dt}\frac{d\eta}{ds} = \frac{1-t}{8} \tag{3}$$

Therefore substitute all information into the integral in (1), it will yielding to;

$$I_2 = Area \int_{-1}^{1} \int_{-1}^{1} f(\frac{(1+s)(1-t)}{4}, \frac{1+t}{2}) \frac{1-t}{8} dsdt$$

$$= Area \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1-t}{8} W_i W_j f(\frac{(1+s)(1-t)}{4}, \frac{1+t}{2}) \tag{4}$$

$$= Area \sum_{r=1}^{nxn} G_r f(u_r, v_r).$$

where G_r denoted the new weights while u_r and v_r denoted the new Gaussian points. Since we had already transform the triangles into square -1 to 1, the Area will be equal to 1.

$$G_r = \frac{1-t}{8} W_i W_j, \qquad u_r = \frac{(1+s)(1-t)}{4}, \qquad v_r = \frac{1+t}{2}. \tag{5}$$

Suppose we want to find Gaussian quadrature of 2×2 points. We eventually will obtain four weights and Gaussian points . We know that, for one-dimension Gaussian quadrature of 2 points as stated in (Teh, 2009) is:

$$\int_{-1}^{1} f(x)dx = f(\frac{-1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) \tag{6}$$

From this one-dimensional Gauss quadrature, we actually extend it into two-dimensional Gauss quadrature where in this case we seek for two-dimensional integral containing s and t which is:

$$\int_{-1}^{1}\int_{-1}^{1} f(s,t)\,dt\,ds = \int_{-1}^{1} F(t)\,ds \qquad (7)$$

where

$$F(t) = \int_{-1}^{1} f(s,t)\,dt$$
$$= [f(s, t = \frac{-1}{\sqrt{3}}) + f(s, t = \frac{1}{\sqrt{3}})]$$

Hence, for two-dimension Gauss quadrature of 2 × 2 points, it will yields to;

$$\int_{-1}^{1}\int_{-1}^{1} f(s,t)\,dt\,ds = \int_{-1}^{1} F(t)\,ds\,dt = f(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}) + f(\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}) \qquad (8)$$

But then, our main intention is to find the Gaussian points and weights as mention in (5), therefore, we substitute the values that we obtain from equation (8) into (5) as below:

for $f(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}})$,

$$G_1 = \frac{(1-\frac{-1}{\sqrt{3}})(1)}{8} = 0.197168783648703,$$

$$u_1 = \frac{(1+\frac{-1}{\sqrt{3}})(1-\frac{-1}{\sqrt{3}})}{4} = 0.166666666666667, \qquad (9)$$

$$v_1 = \frac{1+\frac{-1}{\sqrt{3}}}{2} = 0.211324865405187.$$

for $f(=\frac{1}{\sqrt{3}})$,

$$G_2 = \frac{(1-\frac{1}{\sqrt{3}})(1)}{8} = 0.052831216351297,$$

$$u_2 = \frac{(1+\frac{-1}{\sqrt{3}})(1-\frac{1}{\sqrt{3}})}{4} = 0.044658198738520, \qquad (10)$$

$$v_2 = \frac{1+\frac{1}{\sqrt{3}}}{2} = 0.788675134594813.$$

for $f(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}})$,

$$G_3 = \frac{(1-\frac{-1}{\sqrt{3}})(1)}{8} = 0.197168783648703,$$

$$u_3 = \frac{(1+\frac{1}{\sqrt{3}})(1-\frac{-1}{\sqrt{3}})}{4} = 0.622008467928146, \qquad (11)$$

$$v_3 = \frac{1+\frac{-1}{\sqrt{3}}}{2} = 0.211324865405187.$$

for $f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$,

$$G_4 = \frac{(1 - \frac{1}{\sqrt{3}})(1)}{8} = 0.052831216351297,$$

$$u_4 = \frac{(1 + \frac{1}{\sqrt{3}})(1 - \frac{1}{\sqrt{3}})}{4} = 0.166666666666667, \quad (12)$$

$$v_4 = \frac{1 + \frac{1}{\sqrt{3}}}{2} = 0.788675134594813.$$

We then proceed to find the Gaussian points and weights for r = 2, 3, 4. Not even that, we compute the Gaussian points of 3 × 3 points where, for one-dimension, the Gaussian quadrature as stated in (Teh, 2009), we have;

$$\int_{-1}^{1} f(x)dx = \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right) \quad (13)$$

We continue the computation for Gaussian weights and points for n = 3 where, we use the same steps when finding the higher Gaussian quadrature of 2 points. Two-dimensional integral for standard Gaussian quadrature is stated as in equation (14);

$$\int_{-1}^{1}\int_{-1}^{1} f(s,t)ds\,dt - \frac{25}{81}f\left(-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right) + \frac{40}{81}f\left(-\sqrt{\frac{3}{5}}, 0\right) + \frac{25}{81}f\left(-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right)$$

$$+ \frac{40}{81}f\left(0, -\sqrt{\frac{3}{5}}\right) + \frac{64}{81}f(0,0) + \frac{40}{81}f\left(0, \sqrt{\frac{3}{5}}\right)$$

$$+ \frac{25}{81}f\left(\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right) + \frac{40}{81}f\left(\sqrt{\frac{3}{5}}, 0\right) + \frac{25}{81}f\left(\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right) \quad (14)$$

The illustration of our Gaussian weights and points is stated in Table 1. For next section, we are going to test the Gaussian weights and points over an integral. After that, we will observe the accuracy and the convergence of the numerical results obtained.

Table 1: Points and Weights for higher order Gaussian quadrature for n = 2 and 3 number of points.

n	G	u	v
2	0.197168783648703	0.166666666666667	0.211324865405187
	0.052831216351297	0.044658198738520	0.788675134594813
	0.197168783648703	0.622008467928146	0.211324865405187
	0.052831216351297	0.166666666666667	0.788675134594813
3	0.098765432098765	0.250000000000000	0.500000000000000
	0.013913785849291	0.056350832689629	0.887298334620742
	0.109543004274166	0.443649167310371	0.112701665379258
	0.061728395061728	0.443649167310371	0.500000000000000
	0.008696116155807	0.100000000000000	0.887298334620742
	0.068464377671354	0.787298334620742	0.112701665379258
	0.061728395061728	0.056350832689629	0.500000000000000
	0.008696116155807	0.012701665379258	0.887298334620742
	0.068464377671354	0.100000000000000	0.112701665379258

RESULTS AND DISCUSSION

The efficacy of higher order Gaussian quadrature that we had already computed were tested where, we used all the Gaussian weights and points obtained to an integral containing rational

functions. We consider the integral which contain rational functions due to (Rathod & Karim, 2002). The integral is shown as in equation (15);

$$I_1^{i,j} = \int_0^1 \int_0^{1-y} \frac{x^i y^j}{\alpha + \beta x + \gamma y} dx dy \qquad (15)$$

where i, j are non-negative integer while α, β and γ are constant values. By setting the same value of constant for α and β which is equal to 0.375, we have the following equation:

$$I_1^{k,j} = \int_0^1 \int_0^{1-y} \frac{x^k}{0.375 - 0.375x} dx dy \qquad (16)$$

The convergence of the numerical results obtained and the number of points, n involved is studied with different values of k used. Here, k is the power of the rational functions. Table 2 below show the numerical results computed for $I^{k,0}$ where we consider k = 2, 4 and 6. For the integrand of $\frac{x^k}{\alpha + \beta x + \gamma y}$ with γ = 0, it can be seen that the higher order Gaussian quadrature gave more accurate results compared to the standard Gaussian quadrature.

Table 2: Computed result of integrals in equation (16) for different order of rational functions, k = 2, 4 and 6.

Method	Points	Computed value of $I^{k,0}$		
		k = 2	k = 4	k = 6
Standard GQ	3 × 3	0.444444444444444	0.111111111111111	0.027777777777778
	4 × 4	0.569444444444444	0.213888888888889	0.079567901234568
Higher GQ	3 × 3	0.718531582729114	0.362976027173558	0.210753804951336
	4 × 4	0.784939230000665	0.429383674445109	0.277002722064157
	5 × 5	0.818970941425455	0.463415385869900	0.311034433488948
Exact value		0.888888888888889	0.533333333333333	0.380952380952381

In Table 3, we provide the relative error of the computed results for integral in (16). We find the relative error for the computed results by using the formula, Error= $(E_n - E_a)/E_a$ where E_n and E_a are the numerical and exact values of an integrals respectively. The table illustrate that both methods perform better (the error is decreasing) as the number of points, n used is increasing. As we increase the number of points, n, the results of both approaches will converge to its exact value. However, the relative error for higher order Gaussian quadrature is much more smaller and this depicts that, higher Gaussian quadrature approach is more accurate compared to the standard Gaussian quadrature. Note that, for n = 3, and k = 2, higher Gaussian quadrature show error which is less than 0.5. The order of the rational functions will also effect the convergence of the results obtained. This can be seen from the table, the error increased as the order changed from k = 2 to k = 4 and 6.

Table 3: Computed relative error of integral in equation (16) for different order of rational functions, k = 2, 4 and 6.

Method	Points	Computed relative error of $I^{k,0}$		
		k = 2	k = 4	k = 6
Standard GQ	3 × 3	0.5000	0.7917	0.9271
	4 × 4	0.3594	0.5990	0.7911
Higher GQ	3 × 3	0.1917	0.3194	0.4468
	4 × 4	0.1169	0.1949	0.2729
	5 × 5	0.0787	0.1311	0.1835

Next, we consider the integral where:

$$I^{0,k} = \int_0^1 \int_0^{1-y} \frac{y^k}{0.375 - 0.375y} dx dy$$

Table 4 below show the result of the computed integral of $I^{0,r}$ where in this case we use k = 2, 4 and 6 as in aforementioned integrals. From the table, we can see that when we implement the higher order of Gauss quadrature formula, the numerical evaluation give higher precision in terms of their convergence. Although that, the numerical solution for the standard Gauss quadrature give less accurate solution and low in their convergence rate.

Table 4: Computed results for integral in equation (17) for $k = 2, 4$ and 6

Method	Points	Computed value of $I^{k,0}$		
		k = 2	k = 4	k = 6
Standard GQ	3 × 3	0.444444444444444	0.111111111111111	0.027777777777778
	4 × 4	0.569444444444444	0.213888888888889	0.079567901234568
Higher GQ	3 × 3	0.888888888888889	0.533333333333333	0.380000000000000
	4 × 4	0.888888888888889	0.533333333333333	0.380952380952381
	5 × 5	0.888888888888889	0.533333333333333	0.380952380952381
Exact value		0.888888888888889	0.533333333333333	0.380952380952381

CONCLUSION

We have investigated Gaussian quadrature for higher order implication to the integral containing rational functions. Different approach in order to cope with integral containing rational functions is presented in this paper. We have shown that for increasing number of points, the integral evaluation will eventually lead to the exact values. The evaluation of the integral using higher order Gaussian quadrature provide higher accuracy numerical results compared to standard Gaussian quadrature.

Acknowledgments

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